Exercises # 1: Review of probability theory

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We denote the combination numbers $C_n^p := \frac{n!}{p!(n-p)!} := \frac{n(n-1)\cdots(n-p+1)}{1\cdot 2\cdots p}$.

Exercise 1 (Martin Gardner, *Scientific American* (1959))**.**

(i) *Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?*

(ii) *Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys?*

Solution 1. *In both cases the state space is* $\Omega := \{(G,G),(G,B),(B,G),(BB)\}$ *and all events have the same probability* $\frac{1}{4}$*.*

(i) *Jones family. The event "the older child is a girl" is the event (subset of* Ω*)* $A := \{(G,G),(G,B)\}$*, with* $\mathbb{P}(A) = \frac{2}{4} = \frac{1}{2}$*. The event " both children are girls" is the event* $B := (G,G)$*, with* $\mathbb{P}(B) = \frac{1}{4}$*. Thus the requested probability is*

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)}{\mathbb{P}(A)} = \frac{1}{2}.$$

(ii) *Smith family. The event "at least one of the children is a boy" is the event* $C := \{(B,B),(B,G),(G,B)\}$*, with* $\mathbb{P}(C) = \frac{3}{4}$*. The event " both children are boys" is the event* $D := (B,B)$*, with* $\mathbb{P}(D) = \frac{1}{4}$*. Thus the requested probability is*

$$\mathbb{P}(D|C) = \frac{\mathbb{P}(C \cap D)}{\mathbb{P}(C)} = \frac{\mathbb{P}(D)}{\mathbb{P}(C)} = \frac{1}{3}.$$

Exercise 2. *What is the probability to obtain exactly 3 hearts when drawing 5 cards in a deck of 32 cards (containing exactly 8 hearts) ...*

(i) ... simultaneously?

(ii) ... successively without replacement?

(iii) ... successively with replacement?

Solution 2. (i) *We take here as sample space Ω the set of the 5-element subsets of a set of 32 elements, which contains a total of*

$$card(\Omega) = C_{32}^5 = \frac{32 \times 31 \times 30 \times 29 \times 28}{5 \times 4 \times 3 \times 2} = 8 \times 31 \times 29 \times 28 = 201,376$$

possible cases, all with the same probability. The favorable cases correspond to drawing exactly 3 cards among the 8 hearts and 2 cards among the remaining 24 (which are not hearts). This adds up to

$$C_8^3 C_{24}^2 = \frac{8 \times 7 \times 6}{3 \times 2} \frac{\times 24 \times 23}{2} = 56 \times 12 \times 23 = 15,456$$

favorable cases, i.e. exactly 15,456 hands of 5 cards contain exactly 3 hearts. Thus the probability to obtain exactly 3 hearts is

$$p = \frac{15,456}{201,376} = \frac{56 \times 12 \times 23}{8 \times 31 \times 29 \times 28} = \frac{3 \times 23}{31 \times 29} = \frac{69}{899} = 0.076\ldots$$

(ii) *We take here as sample space Ω the set of sequence of 5 elements drawn without replacement in a set of 32 elements, which contains a total of*

$$card(\Omega) = 32 \times 31 \times 30 \times 29 \times 28 = 24,165,120$$

possible cases, all with the same probability. The favorable cases can be first split according to the location of the 3 hearts, for which there are $C_5^3 = 10$ possibilities. For each of these locations there are $8 \times 7 \times 6$ choices for the heart cards and 24×23 choices for the remaining cards (which are not hearts). This adds up to

$$C_5^3 \times (8 \times 7 \times 6) \times (24 \times 23) = 1,854,720$$

favorable cases, i.e. exactly 1,854,720 sequences of 5 cards drawn successively without replacement contain exactly 3 hearts. Thus the probability to obtain exactly 3 hearts is

$$p = \frac{1,854,720}{24,165,120} = \frac{10 \times (8 \times 7 \times 6) \times (24 \times 23)}{32 \times 31 \times 30 \times 29 \times 28} = \frac{3 \times 23}{31 \times 29} = \frac{69}{899} = 0,076\ldots,$$

which is the same as before.

(iii) We take here as sample space Ω the set of sequence of 5 elements drawn with replacement in a set of 32 elements, which contains a total of

$$card(\Omega) = 32^5 = 33,554,432$$

possible cases, all with the same probability. The favorable cases can be first split according to the location of the 3 hearts, for which there are $C_5^3 = 10$ possibilities. For each of these locations there are 8^3 choices for the heart cards and 24^2 choices for the remaining cards (which are not hearts). This adds up to

$$C_5^3 \times 8^3 \times 24^2 = 2,949,120$$

favorable cases, i.e. 2,949,120 successive draws of 5 cards with replacement contain exactly 3 hearts. Thus the probability to obtain exactly 3 hearts is

$$p = \frac{2,949,120}{33,554,432} = 0.087\ldots$$

This corresponds to a Bernouilli scheme!

Exercise 3. *An urn contains 4 white balls and 3 black balls. You draw 3 balls, one by one, without remise. What is the probability that the first ball is white, the second white and the third black?*

Solution 3. *To make things transparent we identify the balls of the same color with numbers, which doesn't have any impact on probabilities. The seven balls are thus W_1, W_2, W_3, W_4, B_1, B_2, and B_3. We take for sample space Ω the set of sequences of 3 elements drawn without replacement in a universe of 7 elements, which contains a total of*

$$card(\Omega) = A_7^3 \left(= 3!\ C_7^3 \right) = 7 \times 6 \times 5 = 210$$

possible cases, all with the same probability. For favorable cases, there are $4 \times 3 = 12$ *draws of the first two balls such that both are white, and for each of these draws there are 3 possibilities for which the third ball is black. This adds up to* $4 \times 3 \times 3 = 36$ *favorable cases. Thus the requested probability is*

$$\frac{4 \times 3 \times 3}{7 \times 6 \times 5} = \frac{2 \times 3}{7 \times 5} = \frac{6}{35} = 0,17\dots$$

Exercise 4.

1. *State (and prove?) Bayes' formula.*

2. *Mr X has 100 dices among which 25 are* loaded *(unfair). For each piped dice, the probability to obtain a 6 is 0.5.*

   (a) *Mr X draws a randomly selected dice and obtains a 6. What is the probability that this dice is loaded?*

   (b) *Let* $n \in \mathbb{N}^*$ *be a positive integer. Mr X draws* n *times a randomly selected dice and obtains a 6 each time. What is this time the probability* p_n *that this dice is loaded?*

   (c) *Determine* $\lim_{n\to\infty} p_n$*. What does this mean?*

Solution 4.

1. *See textbook.*

2. *(a) Denoting* A *the event "the dice is loaded" and* B *the event "Mr X obtains a the outcome 6", the requested probability is* $\mathbb{P}(A|B)$*. We have* $\mathbb{P}(A) = \frac{25}{100} = \frac{1}{4} \neq 0$ *and* $\mathbb{P}(\bar{A}) = 1 - \frac{1}{4} = \frac{3}{4}$*. Also,* $\mathbb{P}(B|A) = \frac{1}{2}$ *and* $\mathbb{P}(B|\bar{A}) = \frac{1}{6}$*. Hence,*

   $$\mathbb{P}(B) = \mathbb{P}(A) \times \mathbb{P}(B|A) + \mathbb{P}(\bar{A}) \times \mathbb{P}(B|\bar{A}) = \frac{1}{4} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{6} = \frac{1}{4} \neq 0.$$

   From Bayes' formula, the probability that the dice is loaded is

   $$\mathbb{P}(A|B) = \frac{\mathbb{P}(A)\mathbb{P}(B|A)}{\mathbb{P}(A)\mathbb{P}(B|A) + \mathbb{P}(\bar{A})\mathbb{P}(B|\bar{A})} = \frac{1}{2}.$$

   (b) Denoting A *the event "the dice is loaded" and* B_n *the event "Mr X obtain* n *times the outcome 6", the requested probability is* $\mathbb{P}(A|B_n)$*.*

We still have $\mathbb{P}(A) = \frac{1}{4} \neq 0$ *and* $\mathbb{P}(\bar{A}) = \frac{3}{4}$. *Also,* $\mathbb{P}(B_n|A) = \frac{1}{2^n}$ *and* $\mathbb{P}(B_n|\bar{A}) = \frac{1}{6^n}$. *Hence,*

$$\mathbb{P}(B_n) = \mathbb{P}(A) \times \mathbb{P}(B_n|A) + \mathbb{P}(\bar{A}) \times \mathbb{P}(B_n|\bar{A}) = \frac{1}{4} \times \frac{1}{2^n} + \frac{3}{4} \times \frac{1}{6^n} \neq 0.$$

From Bayes' formula, the probability that the dice is loaded is

$$p_n := \mathbb{P}(A|B_n) = \frac{\mathbb{P}(A)\mathbb{P}(B_n|A)}{\mathbb{P}(A)\mathbb{P}(B_n|A) + \mathbb{P}(\bar{A})\mathbb{P}(B_n\bar{|}A)} = \frac{\frac{1}{4} \times \frac{1}{2^n}}{\frac{1}{4} \times \frac{1}{2^n} + \frac{3}{4} \times \frac{1}{6^n}} = \frac{1}{1 + \frac{1}{3^{n-1}}}.$$

(c) $\lim_{n\to\infty} p_n = 1$*: as the intuition suggests, a dice producing repeatedly and indefinitely the same value must be loaded.*

Solution 5.

```
import numpy as np

def roll_a_dice(n = 1, p = 7):
        x = np.random.randint(1, high = p, size = n)
        return x
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