Exercises # 4: Markov Chains

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Each problem is worth 10 points, points above 50 are bonus. Due at the beginning of class on October 4. Python codes must be sent by email before October 4, 12:40pm.

Exercise 1. *Consider a Markov chain with transition matrix*

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

a) *Represent the graph of this Markov chain and determine its communication classes, their nature (recurrent or transient) and their periodicity.*

b) *Determine the stationary probabilities of this Markov chain.*

c) *Is there a limiting distribution?*

d) *How much time does the process spend in average in each of the states (at the limit where the time $n \to \infty$)?*

Exercise 2. *A kangaroo jumps between five points on a circle, At every step he jumps from its location to one of the two neigboring points on the* *circle with probability* 0.5*. Show that the locations of the kangaroo at each step compose a Markov chain and provide its state space, graph and transition matrix; determine the communication classes as well as the period and nature (recurrent or transient) of all states. How much time does the kangaroo spend in average in each of the states (at the limit where the time $n \to \infty$)?*

Exercise 3. *You commute between home and office and you have four umbrellas. Every time, if it rains you take your umbrella; if it doesn't rain you leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, you are at the other, it starts raining and you must leave: in that case, you get wet.*

1. *Show that the number of umbrellas that are in the same location as you are at the times when you need to commute is a Markov chain, draw its graph and give its transition probabilities.*
2. *What are the stationary distributions?*
3. *If the probability of rain is p, what is the probability that you get wet?*
4. *(Bonus) Current estimates show that $p = 0.6$ in East Lansing. How many umbrellas should you have so that, following the strategy described above, the probability that you get wet is less than* 10%*?*

Exercise 4. *Let S_n be a simple (symmetric and one-dimensional) random walk on $\mathbb{Z}$ and let $N := \inf\{n \geq 1 :\}$. What is the expectation of the number of times that S, starting at 0, will visit a given state $i \in \mathbb{Z}, i \neq 0$, before it will come back to 0 for the first time?*

Hint: *The probability that, starting at* 0*, the process will hit i exactly k times before coming back to* 0 *for the first time is the probability that the process will first, starting at* 0*, hit i without hitting* 0*; then, starting at i, return to i without hitting 0, $k-1$ times; and finally, starting at i, hit* 0 *without hitting i.*

Exercise 5. *Consider a gambler who starts with an initial fortune of* $\$x$ *and then places independently successive bets, at each of which he wins or loses* \$1 *with probabilities p and $q := 1-p$, respectively. The gambler stops playing when (if and only if) his fortune reaches* \$0 *or* $\$y$*, $y > x$. Let S_n denote the total fortune after the n^{th} bet, $T(x, y) := \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = y\}$ the time when the game stops, and let $\phi(x) := P(S_T = y)$.*

1. *Show that S is a Markov chain and give its states, transition probabilities, and communication classes as well as classes nature (recurrent or transient) and periodicity.*

2. *Find a recursion on $\phi(x)$.*

3. *If $p = q = \frac{1}{2}$, show that $\phi(x) = \frac{x}{y}$.*

4. *What is $\phi(x)$ when $p \neq q$?*

5. *Does the Markov chain S have any stationary distribution? any limiting distribution?*

6. *Consider the alternative strategy where the gambler's bet all his wealth x at the first bet. Which strategy is best? Does this depend on x, y, or p?*

Exercise 6. *Consider the random walk on the signed integers (as defined in class). Is there any stationary distribution? Is there any limiting distribution?*