Exercises # 5: Applications of Markov Chains

Léo Neufcourt

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Each problem is worth 10 points, points above 40 are bonus. Due at the beginning of class on October 18. Python codes must be sent by email before October 18, 12:40pm.

Exercise 1. *Exercise 63 page 286 in the textbook.*

Solution 1. *The chain has two communication classes,* $T := \{1, 2, 3\}$ *(transient) and* $R := \{4\}$ *(erdodic, absorbing). First,* $s_{i,j} = [(I_T - P_T)^{-1}]_{i,j}$.

$$(I_T - P_T)^{-1} = 10 \begin{bmatrix} \frac{32}{145} & \frac{4}{29} & \frac{9}{145} \\ \frac{14}{145} & \frac{9}{29} & \frac{13}{145} \\ \frac{19}{145} & \frac{6}{29} & \frac{28}{145} \end{bmatrix},$$

so $s_{1,3} = 1000\frac{9}{145} = \frac{18}{29}$, $s_{2,3} = 10\frac{13}{145} = \frac{26}{29}$ *and* $s_{3,3} = 10\frac{28}{145} = \frac{56}{29}$.

For $i, j \in T$ *and* $0 \leq m$, *let* $T_{i,j}(m) := \#\{n \geq m : X_n = j | X_0 = i\}$ *and* $f_{i,j} := P(X_n = j \text{ for some } n \geq 0 | X_0 = i) = P(T_{i,j}(0) > 0)$. *It follows from the Markov property that* $T_{i,j}(0) = T_{i,j}(1) + 1_{i=j}$ *Now we have*

$$\begin{aligned} s_{i,j} &:= \mathbb{E}[T_{i,j}(0)] \\ &= 1_{i=j} + \mathbb{E}[T_{i,j}(1)] \\ &= 1_{i=j} + \mathbb{P}(T_{i,j}(1) > 0)\mathbb{E}[T_{i,j}(1) | T_{i,j}(1) > 0] + \mathbb{P}(T_{i,j}(1) = 0)\mathbb{E}[T_{i,j}(1) | T_{i,j}(1) = 0] \\ &= 1_{i=j} + \mathbb{P}(T_{i,j}(1) > 0)\mathbb{E}[T_{j,j}(1)] \\ &= 1_{i=j} + f_{i,j} s_{j,j} \end{aligned}$$

Thus if $i \neq j$,

$$f_{i,j} = \frac{s_{i,j}}{s_{j,j}};$$

and

$$f_{j,j} = \frac{s_{i,j} - 1}{s_{j,j}}.$$

Hence $f_{1,3} = \frac{9}{28}$, $f_{2,3} = \frac{13}{28}$, *and* $f_{3,3} = \frac{27}{56}$.

Exercise 2. *Exercise 64 page 286 in the textbook.*

Solution 2. *With $\mu < 1$, we have*

$$\begin{aligned} \mathbb{E}[\sum_{k=0}^{\infty} X_k | X_0 = 1] &= \sum_{k=0}^{\infty} \mathbb{E}[X_k | X_0 = 1] \\ &= \sum_{k=0}^{\infty} mu^k \\ &= \frac{1}{1-\mu} \end{aligned}$$

$$\begin{aligned} \mathbb{E}[\sum_{k=0}^{\infty} X_k | X_0 = n] &= n \sum_{k=0}^{\infty} \mathbb{E}[X_k | X_0 = 1] \\ &= \frac{n}{1-\mu} \end{aligned}$$

Exercise 3. *Exercise 66 page 286 in the textbook.*

Solution 3. $\pi_0 := \lim_{n\to\infty} p(X_n = 0)$ *is the smallest root of* $\phi(x) := \sum_{k=0}^{\infty} p_k x^k - x = 0$ *in* $[0, 1]$*, and we know that* 1 *is always a root.*

a) $\phi(x) = \frac{3}{4}x^2 - x + \frac{1}{4} = (x-1)(\frac{3}{4}x - \frac{1}{4})$ *which has roots* $\frac{1}{3}$ *and* 1*, whence* $\pi_0 = \frac{1}{3}$.

b) $\phi(x) = \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4} = \frac{1}{4}(x-1)^2$ *which has double root* 1*, whence* $\pi_0 = 1$.

c) $\phi(x) = \frac{1}{3}x^3 - \frac{1}{2}x + \frac{1}{6} = (x-1)(\frac{1}{3}x^2 + \frac{1}{3}x - \frac{1}{6}) = \frac{1}{6}(x-1)(2x^2 + 2x - 1)$*, which has roots* 1 *and* $\frac{-1\pm\sqrt{3}}{2}$*, whence* $\pi_0 = \frac{\sqrt{3}-1}{2}$.

Exercise 4. *Exercise 67 page 286 in the textbook.*

Solution 4.

a) X *is a Makov chain because the distribution of* X_{n+1} *is fully determined by the composition of the urn after time n, i.e. by* X_n*. The state space is* $\{0, 1, ..., N\}$.

b) Clearly X *is irreducible (each state communicates with its two neighbors) and aperiodic (each state has a positive transition probability to itself). An irreducible Markov chain on a finite state space is always (positive) recurrent.*

c) The non-zero transition probabilities are

- $p_{i,i+1} = p\frac{N-i}{N}$, $0 \le i \le N-1$;
- $p_{i,i-1} = (1-p)\frac{i}{N}$, $1 \le i \le N$;
- $p_{i,i} = p\frac{i}{N} + (1-p)\frac{N-i}{N}$, $0 \le i \le N$.

d) Since states communicate only with their neighbors, solving $\pi P = \pi$ amounts to equating flows $\pi_0 P_{0,1} = \pi_1 P_{1,0}$ and $\pi_1 P_{1,2} = \pi_2 P_{2,1}$, yielding $\pi_0/\pi_1 = \frac{(1-p)/2}{p}$ and $\pi_2/\pi_1 = \frac{p/2}{1-p}$. From $\pi_0+\pi_1+\pi_2 = 1$ we get $\pi_0 = p^2$, $\pi_1 = 2p(1-p)$ and $\pi_2 = (1-p)^2$. The proportion of time spent in each (recurrent) state is precisely given by π.

e) Let us conjecture that $\pi_k = C_N^k p^k (1-p)^{N-k}$.

f) We can prove the conjecture without using the results on time reversible Markov chains. If $1 \leq i \leq N$, we have

$$\begin{aligned}
\sum_i \pi_i p_{i,j} &= \pi_{j-1}p_{j-1,j} + \pi_j p_{j,j} + \pi_{j+1}p_{j+1,j} \\
&= \pi_{j-1}p\frac{N-(j-1)}{N} + \pi_j[p\frac{j}{N} + (1-p)\frac{N-j}{N}] + \pi_{j+1}(1-p)\frac{j+1}{N} \\
&= [p\frac{N-(j-1)}{N}\pi_{j-1} + \pi_j(1-p)\frac{N-j}{N}] + [\pi_j p\frac{j}{N} + \pi_{j+1}(1-p)\frac{j+1}{N}] \\
&= [C_{N-1}^{j-1}p^j(1-p)^{N+1-j} + C_{N-1}^{j}p^j(1-p)^{N+1-j}] \\
&\quad +[C_{N-1}^{j}p^{j+1}(1-p)^{N-j} + C_{N-1}^{j-1}p^{j+1}(1-p)^{N-j}] \\
&= (1-p)C_N^j p^j(1-p)^{N-j} + pC_N^j p^j (1-p)^{N-j} \\
&= \pi_j
\end{aligned}$$

Since the chain is positive recurrent and aperiodic (i.e. ergodic) π is also the limiting distribution.

g) The time T until there are only white balls can be decomposed into $T = \sum_{k=i}^{N-1} T_k$, where T_k is the number of steps needed to go from k to $k+1$ white balls in the urn. T_k is the time of the first success in independent Bernouilli trials, each with success probability $p_k = \frac{N-k}{N}$, so $\mathbb{E}[T_k] = \frac{N}{N-k}$ and $\mathbb{E}[T] = N\sum_{k=1}^{N-i}\frac{1}{k}$.

Exercise 5 (Python). *First or all, simulate a "data" vector x containing 1,000 independent observations of Binomial distribution with $n = 2019$ and $p = 0.5$. This vector $x = (x_1, x_2, ..., x_{1,000})$ is now fixed.*

Now suppose that we have the following model for the data X: first we know that, given the value of a random variable N, $X_1, ..., X_{1,000}$ follow independent Binomial distribution with parameters $n = n$ and $p = 0.5$, i.e.

$$p(X = x|N = n) = \prod_{k=1}^{1,000} C_n^{x_k}\frac{1}{2^n}.$$

Additionally, we also know the general distribution of N: $\mathbb{P}(N = n) = \frac{6}{\pi^2 n^2}$.

Use the Metropolis Hasting algorithm to generate 10,000 samples $(Z_1, Z_2, ..., Z_{10,000}$ *approximately following the conditional distribution* $p(N|X = x)$*, i.e. such that*

$$p(Z_n = j) \approx p(N = j|X = x)$$

when $n \to \infty$*, and plot their histogram. What is (an approximate value of)* $\mathbb{E}[N|X = x]$*? Note that the answer will depend on the values you sampled for* x*.*

You can use numpy, scipy and matplotlib as well as all packages from the standard library.