Exercises # 6: Exponential distribution and Poisson process

Léo Neufcourt

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Each problem is worth 10 points, points above 50 are bonus. Due at the beginning of class on October 28. Python codes must be sent by email before October 28, 12:40pm.

Exercise 1. *Let $p \geq 1$ and $X_1, X_2, ..., X_p$ be independent exponential random variables with respective parameter $\lambda_1, \lambda_2, ..., \lambda_p$. Show that*

(i) $\min_{k=1,...,p} X_k \sim \mathcal{E}xp(\sum_{k=1}^{p} \lambda_k)$;

(ii) $\forall 1 \leq i \leq p, \mathbb{P}(\min_{k=1,...,p} X_k = X_i) = \frac{\lambda_i}{\sum_{k=1}^{p} \lambda_k}$;

(iii) The rank ordering of the X_ks is independent from the value of $\min_{k=1,...,p} X_k$.

Exercise 2. *You are receiving rewards at a stochastic rate $R_\lambda(t)$ until some exponential random time $T \geq 0$ with parameter $\lambda > 0$. We suppose that T is independent of the reward rate R_λ (given λ), i.e.*

$$\mathbb{P}(T \leq u, R_\lambda(t_1) \leq r_1, ..., R_\lambda(t_k) \leq r_k) = \mathbb{P}(T \leq u)\mathbb{P}(R_\lambda(t_1) \leq r_1, ..., R_\lambda(t_k) \leq r_k)$$

for all $\lambda > 0$, $k \geq 1$, $u, t_1, t_2, ..., t_k \geq 0$, $r_1, r_2, ..., r_k \in \mathbb{R}$. However, R_λ may also depends on λ.

1. *Show that $\mathbb{E}[\int_0^T R_\lambda(t)dt] = \int_0^\infty e^{-\lambda t}\mathbb{E}[R_\lambda(t)]dt$, for every $\lambda > 0$.*

2. *We suppose that the reward rate R_λ also depends on $\lambda > 0$, in the following way:*

$$R_\lambda(t) \sim \mathcal{N}\big(e^{t\ln(\lambda)}, e^{-127t}\frac{t^{314}}{2019}\big).$$

How would you choose λ in order to maximize the expected reward $\mathbb{E}[\int_0^T R_\lambda(t)dt]$? Remembering that $\mathbb{E}[e^{uT}] = \frac{\lambda}{\lambda - u}$ for every $u < \lambda$ may shorten some calculations.

A possible interpretation of this problem is the following: you can choose investment strategies with risk increasing with λ; riskier strategies lead to a higher reward rate in average, but also an earlier default time.

Exercise 3. *Show that a counting process N starting at 0 has independent increments if and only if $N_t - N_s$ and N_s are independent for every $s \leq t$.*

Exercise 4. *Let N be a Poisson process. What is the limit $\lim_{t\to\infty} \frac{N_t}{t}$?*

Exercise 5. *Show that for a Poisson process N with rate parameter λ, we have*

$$\mathbb{P}(N_t \text{ is even}) = e^{-\lambda t}\cosh(\lambda t)$$

for every $t \geq 0$. What is $\mathbb{P}(N_t \text{ is odd})$?

Exercise 6 (Python)**.** *Simulate 10 paths of a Poisson process with parameter 1 on the time interval $[0, 365]$ and plot them on the same figure. (Hint: starting from exponential interarrival times is probably the easiest way.)*

Simulate another 10 paths of a Poisson process with parameter 0.1 on the time interval $[0, 365]$ and plot them on a second figure. You can appreciate the differences, keeping in mind the result of Exercise 4.