Exercises # 6: Exponential distribution and Poisson process

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Each problem is worth 10 points, points above 50 are bonus. Due at the beginning of class on October 28. Python codes must be sent by email before October 28, 12:40pm.

Exercise 1. *Let $p \geq 1$ and $X_1, X_2, ..., X_p$ be independent exponential random variables with respective parameter $\lambda_1, \lambda_2, ..., \lambda_p$. Show that*

(i) $\min_{k=1,...,p} X_k \sim \mathcal{E}xp(\sum_{k=1}^p \lambda_k)$;

(ii) $\forall 1 \leq i \leq p, \mathbb{P}(\min_{k=1,...,p} X_k = X_i) = \frac{\lambda_i}{\sum_{k=1}^p \lambda_k}$;

(iii) The rank ordering of the X_ks is independent from the value of $\min_{k=1,...,p} X_k$.

Solution 1. *See the textbook for these classical results.*

Exercise 2. *You are receiving rewards at a stochastic rate $R_\lambda(t)$ until some exponential random time $T \geq 0$ with parameter $\lambda > 0$. We suppose that T is independent of the reward rate R_λ (given λ), i.e.*

$$\mathbb{P}(T \leq u, R_\lambda(t_1) \leq r_1, ..., R_\lambda(t_k) \leq r_k) = \mathbb{P}(T \leq u)\mathbb{P}(R_\lambda(t_1) \leq r_1, ..., R_\lambda(t_k) \leq r_k)$$

for all $\lambda > 0$, $k \geq 1$, $u, t_1, t_2, ..., t_k \geq 0$, $r_1, r_2, ..., r_k \in \mathbb{R}$. However, R_λ may also depends on λ.

1. *Show that $\mathbb{E}[\int_0^T R_\lambda(t)dt] = \int_0^\infty e^{-\lambda t}\mathbb{E}[R_\lambda(t)]dt$, for every $\lambda > 0$.*

2. *We suppose that the reward rate R_λ also depends on $\lambda > 0$, in the following way:*

$$R_\lambda(t) \sim \mathcal{N}\big(e^{t\ln(\lambda)}, e^{-127t}\frac{t^{314}}{2019}\big).$$

How would you choose λ in order to maximize the expected reward $\mathbb{E}[\int_0^T R_\lambda(t)dt]$? Remembering that $\mathbb{E}[e^{uT}] = \frac{\lambda}{\lambda-u}$ for every $u < \lambda$ may shorten some calculations.

A possible interpretation of this problem is the following: you can choose investment strategies with risk increasing with λ; riskier strategies lead to a higher reward rate in average, but also an earlier default time.

Solution 2. *1.*

$$\begin{aligned}
\mathbb{E}[\int_0^T R_\lambda(t)dt] &= \mathbb{E}[\int_0^\infty 1_{t\leq T} R_\lambda(t)dt] \\
&= \int_0^\infty \mathbb{E}[1_{t\leq T} R_\lambda(t)]dt \\
&= \int_0^\infty \mathbb{E}[1_{t\leq T}]\mathbb{E}[R_\lambda(t)]dt \\
&= \int_0^\infty \mathbb{P}(1_{t\leq T})\mathbb{E}[R_\lambda(t)]dt \\
&= \int_0^\infty e^{-\lambda t}\mathbb{E}[R_\lambda(t)]dt
\end{aligned}$$

2. Let $u := \log(\lambda)$, so that we have

$$\mathbb{E}[\int_0^T R_\lambda(t)dt] = \int_0^\infty e^{-\lambda t} e^{\log(\lambda)t} dt = \frac{1}{\lambda}\mathbb{E}[e^{uT}]$$

Follow

$$\mathbb{E}[\int_0^T R_\lambda(t)dt] = \frac{1}{\lambda - \log(\lambda)}$$

which is maximal at $\lambda = 1$. Of course the strange variance of $R_\lambda(t)$ plays no role here.

Exercise 3. *Show that a counting process N starting at 0 has independent increments if and only if $N_t - N_s$ and N_s are independent for every $s \leq t$.*

Solution 3. *If N has independent increments it follows from the definition that $N_t - N_s$ is independent from $N_s = N_s - N_0$ for every $s \leq t$. Conversely, suppose that $N_t - N_s$ is independent from $N_s = N_s$ for every $s \leq t$ and let $s \leq t \leq u \leq v$. Since $N_v - N_u = (N_v - N_t) - (N_u - N_t)$, $N_v - N_u$ is independent from N_t. Also, $N_v - N_u = (N_v - N_s) - (N_u - N_s)$, so $N_v - N_u$ is independent from N_s. Thus $N_t - N_u$ is independent from $N_t - N_s$.*

Exercise 4. *Let N be a Poisson process. What is the limit $\lim_{t\to\infty} \frac{N_t}{t}$?*

Solution 4. *When t is an integer, we can decompose $N_t = \frac{1}{t}\sum_{k=1}^{t}(N_t - N_{t-1})$ as the sample average of i.i.d. Poisson variables with parameter λ and converge to λ a.s. when $t \to \infty$ ($t \in \mathbb{N}$) according to the (strong) law of large numbers.*

Now for general $t \geq 1$, denoting $[t]$ the integer part of t, we have $[t] \leq t \leq [t]+1$ and $\frac{1}{[t]+1} \leq \frac{1}{t} \leq \frac{1}{[t]}$. Also N being non decreasing implies $N_{[t]} \leq N_t \leq N_{[t]+1}$. Thus:

$$\frac{[t]}{[t]+1}\frac{N_{[t]}}{[t]} = \frac{N_{[t]}}{[t]+1} \leq \frac{N_t}{t} \leq \frac{N_{[t]+1}}{[t]} = \frac{[t]+1}{[t]}\frac{N_{[t]+1}}{[t]+1},$$

It is clear from the case where t is an integer that both bounds converge to λ as $t \to \infty$, and it is well known that if two policemen are escorting a prisoner between them, and both officers go to a cell λ, then the prisoner must also end up in the cell λ.

Exercise 5. *Show that for a Poisson process N with rate parameter λ, we have*

$$\mathbb{P}(N_t \text{ is even}) = e^{-\lambda t}\cosh(\lambda t)$$

for every $t \geq 0$, where $\cosh(x) := \frac{e^x+e^{-x}}{2}$. What is $\mathbb{P}(N_t \text{ is odd})$?

Solution 5.

$$P(N_t = 2n) = e^{-\lambda t}\frac{(\lambda t)^{2n}}{(2n)!}$$

so

$$\mathbb{P}(N_t \text{ is even}) = e^{-\lambda t}\sum_n \frac{(\lambda t)^{2n}}{(2n)!}$$

This is the sum of the even rank terms of the exponential series, which can be obtained by summing the exponential of opposite numbers: $e^x = \sum_n \frac{x^n}{n!}$ implies

$$\cosh(x) := \frac{e^x + e^{-x}}{2} = \sum_n \frac{x^n + (-x)^n}{2n!} = \sum_n \frac{x^{2n}}{(2n)!},$$

Follow the claim: $\mathbb{P}(N_t \text{ is even}) = e^{-\lambda t}\cosh(\lambda t)$.

Letting $\sinh(x) := \frac{e^x - e^{-x}}{2}$ it is easy to check that $\cosh + \sinh = e^x$, from which $e^{-x}\cosh(x) + e^{-x}\sinh = 1$ and

$$\mathbb{P}(N_t \text{ is odd}) = 1 - \mathbb{P}(N_t \text{ is even}) = 1 - e^{-\lambda t}\cosh(\lambda t) = e^{-\lambda t}\sinh(\lambda t)$$

Exercise 6 (Python)**.** *Simulate 10 paths of a Poisson process with parameter 1 on the time interval $[0, 365]$ and plot them on the same figure. (Hint: starting from exponential interarrival times is probably the easiest way.)*

Simulate another 10 paths of a Poisson process with parameter 0.1 on the time interval $[0, 365]$ and plot them on a second figure. You can appreciate the differences, keeping in mind the result of Exercise 4.