Exercises # 7: Poisson process: application and generalizations

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Each problem is worth 10 points, please do all problems. Due at the beginning of class on **Wednesday, November 6**. Python codes must be sent by email before Wednesday, November 6, 12:40pm.

Exercise 1. *Suppose that the number of customer entering a bank follows a Poisson process with rate parameter 3 (per hour).*

(i) *What is the expected time before the first customer enters the bank?*

(ii) *Given that exactly 10 customers have entered the shop within the first hour, what is the expected time at which the first customer entered the bank?*

(iii) *Given that exactly 5 customers have entered the shop within the first opening hour, what is the probability that the last customer arrived in the last five minutes?*

Exercise 2.

1. *Let N be a (homogeneous) Poisson process with rate parameter λ. Show that, for every fixed $s \geq 0$, $(N_{t+s} - N_s)$ is a (homogeneous) Poisson process with rate parameter λ.*

2. *Let N be a non-homogeneous Poisson process with a (deterministic) intensity function $t \mapsto \lambda(t)$. Under which condition(s) on the function λ does the process N have stationary increments? (In other words, for which rate function λ do N_t and $N_{s+t} - Ns$ have the same distribution for every $s, t \geq 0$?)*

Exercise 3. *Please solve Exercise 71 on page 365 of the textbook.*

Exercise 4. *A casino offers the following game, for a fee of* $\$1$*: at time* $t = 0$*, the player starts with a score* 0*, and a Poisson process* N *with rate parameter 1 (per minute) is started; with* $S_0 := 0$*, the jump times* S_i *of* N *(i.e. the time at which the events canonically associated with* N *occur) serve as a random clock for the game: at every even jump time* S_i*, the player's score increases by* 1 *if the time* $\tau_i := S_i - S_{i-1}$ *elapsed since the preceding jump time is larger than* $1s$ *else the score doesn't change). At the end of the game, the player receives his or her score in \$ amount.*

1. *Suppose that the game stops at time* $T = 1$ *minute. Do you want to play this game?*
2. *Suppose that the game stops at time* $T = 5$ *minutes. Do you want to play this game?*
3. *Suppose that the game stops at a fixed time* T*; what would be the fair price for playing this game?*
4. *Now suppose that the game stops when your score reaches 2 or* N *reaches 5, whichever comes first. Do you want to play this game?*

Exercise 5.

1. *Let* N *be a non-homogeneous Poisson process with intensity function* $\lambda(t) := t$.
   - *(i) What is* $\mathbb{E}[N_t]$*?*
   - *(ii) What is* $\mathbb{V}ar[N_t]$*?*
   - *(iii) What is* $\lim_{t\to\infty} \frac{N_t}{t}$*?*
   - *(iv) What is* $\lim_{t\to\infty} \frac{N_t}{t^2}$*?*
   - *(v) What is* $\lim_{t\to\infty} \frac{N_t}{t^3}$*?*

   Hint : *recall that we have shown* $\frac{Poisson(\lambda t)}{t} \xrightarrow[t\to\infty]{} \lambda$ *a.s., and thus in probability and in distribution as well, for any non-negative real number* λ.
2. *Let* N *be a non-homogeneous Poisson process with intensity function* $\lambda(t) = e^{-t}$.
   - *(i) What is* $\mathbb{E}[N_t]$*?*
   - *(ii) What is* $\mathbb{V}ar[N_t]$*?*
   - *(iii) What is* $\mathbb{P}(N_t = 1)$*?*
   - *(iv) What is* $\lim_{n\to\infty} \mathbb{P}(N_t = 1)$*?*