# Exercises # 7: Poisson process: application and generalizations

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Each problem is worth 10 points, please do all problems. Due at the beginning of class on **Wednesday, November 6**. Python codes must be sent by email before Wednesday, November 6, 12:40pm.

**Exercise 1.** *Suppose that the number of customer entering a bank follows a Poisson process $N$ with rate parameter 3 (per hour).*

*(i) What is the expected time before the first customer enters the bank?*

*(ii) Given that exactly 10 customers have entered the shop within the first hour, what is the expected time at which the first customer entered the bank?*

*(iii) Given that exactly 5 customers have entered the shop within the first opening hour, what is the probability that the last customer arrived in the last five minutes?*

**Solution 1.**

*(i) The time before the first customer arrives follows an exponential distribution with parameter $\lambda = 3$ whence mean $\frac{1}{3}$ hour or 20 minutes.*

*(ii) Conditionally to $N_1 = 10$, the first event (customer entering the bank) is distributed as the minimum of 10 independent uniform random variables $U_i$ on $(0, 1)$, which has mean*

$$\begin{aligned}\int_0^1 P(U_i > x, i = 1, ..., 10)dx &= \int_0^1 P(U_i > x)^{10}dx \\ &= \int_0^1 (1-x)^{10}dx \\ &= \frac{1}{11}\end{aligned}$$

*hour or approximately 5 minutes and 27 seconds.*

*(iii)* *Conditionally to* $N_1 = 5$, *the last event (customer entering the bank) is distributed as the maximum of 5 independent uniform random variables* $U_i$ *on* $(0,1)$, *whence the probability that is is larger that 5 minutes is*

$$\begin{aligned} P(\max U_i \geq \frac{55}{60}) &= P(\max U_i \geq \frac{11}{12}) \\ &= 1 - P(\max U_i \leq \frac{11}{12}) \\ &= 1 - P(U_i \leq \frac{11}{12}, i = 1, ..., 5) \\ &= 1 - P(U_i \leq \frac{11}{12})^5 \\ &= 1 - \frac{11}{12}^5 \\ &\approx 0.3527 \end{aligned}$$

**Exercise 2.**

1. *Let* $N$ *be a (homogeneous) Poisson process with rate parameter* $\lambda$. *Show that, for every fixed* $s \geq 0$, $(N_{t+s} - N_s)$ *is a (homogeneous) Poisson process with rate parameter* $\lambda$.

2. *Let* $N$ *be a non-homogeneous Poisson process with a (deterministic) intensity function* $t \mapsto \lambda(t)$. *Under which condition(s) on the function* $\lambda$ *does the process* $N$ *have stationary increments? (In other words, for which rate function* $\lambda$ *do* $N_t$ *and* $N_{s+t} - N_s$ *have the same distribution for every* $s, t \geq 0$*?)*

**Solution 2.**

1. *Let* $\widetilde{N}_t := N_{s+t} - N_s$. *Clearly,* $\widetilde{N}_0 = 0$ *and* $\widetilde{N}$ *has independent increments; additionally* $\widetilde{N}_v - \widetilde{N}_u \sim \mathcal{P}(\lambda \times (v - u)$ *for every* $u \leq v$; *this characterizes a Poisson process.*

2. *For any* $s, t \geq s$, $N_{s+t} - N_s \sim \mathcal{P}(\Lambda(s,t))$, *where* $\Lambda(s,t) := \int_s^{s+t} \lambda(u)du = \int_0^t \lambda(s+u)du$. *In particular, the mean of* $N_{s+t} - N_s$ *is given by* $\Lambda(s,t)$ *Stationary increments means that* $\Lambda(s,t)$ *in independent of* $s$; *since* $\Lambda(s,t)$ *is differentiable with respect to* $s$, *its derivative* $\partial_s \Lambda(s,t) = \lambda(s+t) - \lambda(s)$ *must equal 0 for every* $s, t \geq 0$, *whence* $\lambda$ *must be constant and* $N$ *must be a "true" Poisson process So, a non-homogeneous Poisson process with stationary increments is a (homogeneous) Poisson process.*

**Exercise 3.** *Please solve Exercise 71 on page 365 of the textbook.*

**Solution 3.** *Conditionally to* $N_t = n$, $S_1, ..., S_n$ *have the same (joint) distribution as the order statistics* $U_{(1)}, U_{(2)}, ..., U_{(n)}$ *of i.i.d. random variables* $U_1, U_2, ..., U_n$ *uniformly distributed on* $(0, t)$. *It follows that for every* $x \in \mathbb{R}$

$$\begin{aligned}
\mathbb{P}[\sum_{i=1}^{N_t} g(S_i) > x | N_t = n] &= \mathbb{P}[\sum_{i=1}^{N_t} g(U_{(i)}) > x | N_t = n] \\
&= \mathbb{P}[\sum_{i=1}^{n} g(U_{(i)}) > x] \\
&= \mathbb{P}[\sum_{i=1}^{n} g(U_i) > x] \\
&= \mathbb{P}[\sum_{i=1}^{n} g(U_i) > x] \\
&= \mathbb{P}[\sum_{i=1}^{N_t} g(U_i) > x | N_t = n]
\end{aligned}$$

*and*

$$\begin{aligned}
\mathbb{P}[\sum_{i=1}^{N_t} g(S_i) > x] &= \sum_{n \geq 0} \mathbb{P}(N_t = n) \mathbb{P}[\sum_{i=1}^{N_t} g(S_i) > x | N_t = n] \\
&= \sum_{n \geq 0} \mathbb{P}(N_t = n) \mathbb{P}[\sum_{i=1}^{N_t} g(U_i) > x | N_t = n] \\
&= \mathbb{P}[\sum_{i=1}^{N_t} g(U_i) > x].
\end{aligned}$$

*In particular,*

$$\mathbb{E}[\sum_{i=1}^{N_t} g(S_i)] = \mathbb{E}[\sum_{i=1}^{N_t} g(U_i)] = \mathbb{E}[N_t]\mathbb{E}[g(U_1)] = \lambda t \int_0^t g(x) \frac{dx}{t} = \lambda \int_0^t g(x) dx$$

$$\mathbb{V}ar[\sum_{i=1}^{N_t} g(S_i)] = \mathbb{V}ar[\sum_{i=1}^{N_t} g(U_i)] = \mathbb{E}[N_t]\mathbb{E}[g(U_1)^2] = \lambda t \int_0^t g(x)^2 \frac{dx}{t} = \lambda \int_0^t g(x)^2 dx$$

**Exercise 4.** *A casino offers the following game, for a fee of \$1: at time $t = 0$, the player starts with a score 0, and a Poisson process $N$ with rate parameter 1 (per minute) is started; with $S_0 := 0$, the jump times $S_i$ of $N$ (i.e. the time at which the events canonically associated with $N$ occur) serve as a random clock for the game: at every even jump time $S_i$, the player's score increases by 1 if the time $\tau_i := S_i - S_{i-1}$ elapsed since the preceding jump time is larger than $1s$ else the score doesn't change. At the end of the game, the player receives his or her score in \$ amount.*

1. *Suppose that the game stops at time $T = 1$ minute. Do you want to play this game?*
2. *Suppose that the game stops at time $T = 5$ minutes. Do you want to play this game?*
3. *Suppose that the game stops at a fixed time $T$; what would be the fair price for playing this game?*
4. *Now suppose that the game stops when your score reaches 2 or $N$ reaches 5, whichever comes first. Do you want to play this game?*

**Solution 4.** *Let ${X'}_t$ denote the score at time t and denote as well $X_t$ the score obtained in the (more advantageous) variant where the score is increased by 1 at both even and odd jumps of $N$ for which the time elapsed since the previous jump is larger than $1s$.*

1. *The score $X_1$ at time 1 is always lower or equal to $N_1$, and $\mathbb{E}[N_1] = 1$; since $P(X_1 < N_1) > P(0 < S_1 < \frac{1}{60} < 1 < S_2) > 0$, $\mathbb{E}[X_1] < \$1$ and*

$$\mathbb{E}[{X'}_1] \leq \mathbb{E}[X_1] < \$1$$

   *(the game is disadvantageous in average).*

2. *Consider the following five i.i.d. events: $N$ has exactly zero event on $(i, i + \frac{1}{60})$ and exactly one event on $(i+\frac{1}{60}, i+1)$, $i = 0, ..., 4$. each of which has a probability $q := e^{-\frac{1}{60}} e^{-\frac{59}{60}} \frac{59}{60} \approx 0.37$. Thus the expected score $\mathbb{E}[X_t]$ is larger than the binomial expectation $5q$, i.e.*

$$\mathbb{E}[X_t] \geq 5q \approx \$1.87 > \$1.$$

   *Consider the following five i.i.d. events: $N$ has exactly zero event on $(i, i + \frac{1}{60})$, exactly one event on $(i+\frac{1}{60}, i+\frac{1}{2})$, exactly zero event on $(i+\frac{1}{2}, i+1)$, $i = 0, 1, 2, 3, 4$, each of which has a probability $q' := e^{-\frac{1}{60}} \times e^{-\frac{29}{60}} \frac{29}{60} \times \frac{1}{2} e^{-\frac{1}{2}} \approx 0.24$. Thus the expected score $\mathbb{E}[{X'}_T]$ is larger than the binomial expectation $5q$, i.e.*

$$\mathbb{E}[X_T] \geq \mathbb{E}[{X'}_T] \geq 5q' \approx \$1.21 > \$1.$$

3. *We have seen in (i) and (ii) that the expected score satisfies the bounds:*

$$q'T \leq \mathbb{E}[X'_t] \leq \mathbb{E}[X_T] \leq T.$$

*To get the exact expectations, suppose first that $N_t = n$; in that case, the times are distributed as the order statistics of i.i.d. uniform random variables $U_i$ on $(0,t)$. Thus conditionally to $N_t = n$, the expected scores are*

$$\begin{aligned}
E[X_T|N_t = n] &= E[\sum_{i=1}^{n} 1_{S_i - S_{i-1} > \frac{1}{60}} | N_t = n] \\
&= nE[1_{S_1 > \frac{1}{60}} | N_t = n] \\
&= nP[S_1 > \frac{1}{60} | N_t = n] \\
&= nP[U_1 > \frac{1}{60}]^n \\
&= n(1 - \frac{1}{60t})^n,
\end{aligned}$$

$$\begin{aligned}
E[X_T|N_t = 2n] &= E[\sum_{i=1}^{n} 1_{S_{2i} - S_{2i-1} > \frac{1}{60}} | N_t = 2n] \\
&= nE[1_{S_1 > \frac{1}{60}} | N_t = 2n] \\
&= nP[S_1 > \frac{1}{60} | N_t = 2n] \\
&= nP[U_1 > \frac{1}{60}]^{2n} \\
&= n(1 - \frac{1}{60t})^{2n}
\end{aligned}$$

*and*

$$\begin{aligned}
E[X_T|N_t = 2n+1] &= E[\sum_{i=1}^{n} 1_{S_{2i} - S_{2i-1} > \frac{1}{60}} | N_t = 2n+1] \\
&= nE[1_{S_1 > \frac{1}{60}} | N_t = 2n+1] \\
&= nP[S_1 > \frac{1}{60} | N_t = 2n+1] \\
&= nP[U_1 > \frac{1}{60}]^{2n+1} \\
&= n(1 - \frac{1}{60t})^{2n+1}
\end{aligned}$$

*Finally the expected scores are*

$$E[X_T] = \sum_{n=1}^{\infty} P(N_t = n)E[X_T|N_t = n] = \sum_{n=1}^{\infty} e^{-t}\frac{t^n}{n!}n(1 - \frac{1}{60t})^n = (t - \frac{1}{60})e^{-\frac{1}{60}}$$

*and*

$$\begin{aligned}
E[X'_T] &= \sum_{n=1}^{\infty} P(N_t = n) E[X'_T | N_t = n] \\
&= \sum_{n=1}^{\infty} P(N_t = 2n) E[X'_T | N_t = 2n] + \sum_{n=1}^{\infty} P(N_t = 2n+1) E[X'_T | N_t = 2n+1] \\
&= \sum_{n=1}^{\infty} e^{-t} \frac{t^{2n}}{2n!} n (1 - \frac{1}{60t})^{2n} + \sum_{n=1}^{\infty} e^{-t} \frac{t^{2n+1}}{(2n+1)!}) n (1 - \frac{1}{60t})^{2n+1} \\
&= \frac{1}{2}(t - \frac{1}{60}) e^{-t} sh(t - \frac{1}{60}) + \frac{1}{2} e^{-t} [(t - \frac{1}{60})(ch(t - \frac{1}{60}) - 1) - sh(t - \frac{1}{60})] \\
&= \frac{1}{2} e^{-t} [(t - \frac{1}{60})(ch(t - \frac{1}{60}) + sh(t - \frac{1}{60}) - 1) - sh(t - \frac{1}{60})] \\
&= \frac{1}{2} e^{-t} [(t - \frac{1}{60})(e^{t - \frac{1}{60}} - 1) - sh(t - \frac{1}{60})]
\end{aligned}$$

*For $T = 1$, this gives:*

$$E[X_T] = \frac{59}{60} e^{-\frac{1}{60}} \approx \$0.97$$

$$E[X'_T] = \frac{e^{-1}}{2} \frac{59}{60} [(e^{\frac{59}{60}} - 1) - sh(\frac{59}{60}) \approx \$0.09.$$

*With $T = 5$, this gives*

$$E[X_T] = \frac{299}{60} e^{-\frac{1}{60}} \approx \$4.9$$

$$E[X'_T] = \frac{e^{-5}}{2} (\frac{299}{60}) [(e^{\frac{299}{60}} - 1) - sh(\frac{299}{60}) \approx \$2.18.$$

4. *First consider the variant where the score is not bounded by 2, i.e. the score is simply the score at time 5, and denote $\widetilde{X}_t$ the score at time $t$ for $t = 1, ..., 5$. The interarrival times $\tau_i := S_i - S_{i-1}, i = 1, ..., 5$, are i.i.d. exponential random variables, each of which is larger than $\frac{1}{60}$ with probability $e^{-\frac{1}{60}}$. Thus $\widetilde{X}_5$ is simply the number of success in independent trials, each of which has success probability $e^{-\frac{1}{60}}$: $\widetilde{X}_5$ follows a binomial distribution $\mathcal{B}(n = 5, p = e^{-\frac{1}{60}})$. In particular,*

$$\pi_0 := P(\widetilde{X}_5 = 0) = (1 - e^{-\frac{1}{60}})^5 \approx 1.2310^{-9}$$

$$\pi_1 := P(\widetilde{X}_5 = 1) = 5(e^{-\frac{1}{60}})(1 - e^{-\frac{1}{60}})^4 \approx 3.6710^{-7}.$$

*Second, the actual score $X_T$ is given by $X_T := \min(\widetilde{X}_5, 2)$, so that $X_T = 0$ with probability $\pi_0$, $X_T = 1$ with probability $\pi_1$, and $X_T = 2$ with probability $1 - \pi_0 - \pi_1$; in particular*

$$\mathbb{E}[X_T] = \pi_1 + 2(1 - \pi_0 - \pi_1) = 2 - 2\pi_0 - \pi_1 \approx \$2 > \$1$$

*Finally, it is easy to see that* $X'_T \leq \frac{X_t}{2}$*, which implies*

$$\mathbb{E}[X'_T] \leq \frac{\mathbb{E}[X_T]}{2} = \frac{2 - 2\pi_0 - \pi_1}{2} < \$1$$

**Exercise 5.**

1. *Let* $N$ *be a non-homogeneous Poisson process with intensity function* $\lambda(t) := t$.

   (i) *What is* $\mathbb{E}[N_t]$*?*

   (ii) *What is* $\mathbb{V}ar[N_t]$*?*

   (iii) *What is* $\lim_{t\to\infty} \frac{N_t}{t}$*?*

   (iv) *What is* $\lim_{t\to\infty} \frac{N_t}{t^2}$*?*

   (v) *What is* $\lim_{t\to\infty} \frac{N_t}{t^3}$*?*

   Hint : *recall that we have shown* $\frac{Poisson(\lambda t)}{t} \xrightarrow[t\to\infty]{} \lambda$ *a.s., and thus in probability and in distribution as well, for any non-negative real number* $\lambda$.

2. *Let* $N$ *be a non-homogeneous Poisson process with intensity function* $\lambda(t) = e^{-t}$.

   (i) *What is* $\mathbb{E}[N_t]$*?*

   (ii) *What is* $\mathbb{V}ar[N_t]$*?*

   (iii) *What is* $\mathbb{P}(N_t = 1)$*?*

   (iv) *What is* $\lim_{t\to\infty} \mathbb{P}(N_t = 1)$*?*

**Solution 5.**

1. *Let* $N$ *be a non-homogeneous Poisson process with intensity function* $\lambda(t) := t$.

   (i) $\mathbb{E}[N_t] = \int_0^t s ds = \frac{t^2}{2}$. *Thus* $N_t$ *is a Poisson random variable with parameter* $\frac{t^2}{2}$*, and*

   (ii) $\mathbb{V}ar[N_t] = \mathbb{E}[N_t] = \frac{t^2}{2}$*?*

   3-5 *We showed that a "true" Poisson process* $\widetilde{N}_t$ *satisfies* $\frac{\widetilde{N}_t}{t} \xrightarrow[t\to\infty]{a.s.} \lambda$. *Thus, if* $\widetilde{N}_t \sim \mathcal{P}(\lambda t)$*, then*

   $$\frac{\widetilde{N}_t}{\lambda t} \xrightarrow[t\to\infty]{d,P} 1.$$

   *Since* $N_t \sim \mathcal{P}(\frac{t^2}{2})$*, we have here* $\frac{N_t}{\frac{t^2}{2}} \xrightarrow[t\to\infty]{d,P} 1$*, from which follow in probability*

   $$\lim_{t\to\infty} \frac{N_t}{t} = \infty, \lim_{t\to\infty} \frac{N_t}{t^2} = \frac{1}{2}, \lim_{t\to\infty} \frac{N_t}{t^3} = 0.$$

2. *Let $N$ be a non-homogeneous Poisson process with intensity function $\lambda(t) = e^{-t}$.*

(i) $\mathbb{E}[N_t] = \int_0^t e^{-s} ds = 1 - e^{-t}$.

*Thus $N_t$ is a Poisson random variable with parameter $1 - e^{-t}$, and ...*

(ii) ... $\mathbb{V}ar[N_t] = \mathbb{E}[N_t] = 1 - e^{-t}$.

(iii) ... $\mathbb{P}(N_t = 1) = (1 - e^{-t})e^{-(1-e^{-t})}$

(iv) $\lim_{t\to\infty} \mathbb{P}(N_t = 1) = e^{-1}$.

*In fact,* $\lim_{t\to\infty} \mathbb{P}(N_t = n) = \frac{e^{-n}}{n!}$, *which means that $N$ converges (in distribution) to a Poisson distribution with parameter 1.*