Exercises # 8: Continuous-time Markov chains

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Due at 12:40pm on **Friday, November 15**. Python codes must be sent by email before due time. Each problem is worth 10 points, please do all problems.

Exercise 1. *Give an example of a continuous-time Markov chain X with more than one state, and explain why it is a continuous-time Markov chain. What is the expected time before the first transition of X occur? What is the transition matrix Q associated with X? (Recall that $q_{i,j}$ is the probability that X, starting in state i, will make its next transition to state j). What is the asymptotic behavior of the process (long-run proportion of time spent in each state, limiting distribution)?*

Bonus: write a Python code to simulate 10 paths of this continuous time Markov chain on a well-chosen time interval, and plot them on the same figure. Plot also as a function of time the proportions of time spent in each state.

Exercise 2. *Please do Problem 5 page 412 in the textbook.*

Exercise 3. *Please do Problem 9 page 413 in the textbook.*

Exercise 4. *Customers and taxis arrive to a taxi station according to independent Poisson processes with respective rates of two and three per minute. Taxis wait regardless of the number of other taxis present. However, a customer who does not find any taxi waiting when he arrives leaves immediately. What are:*

(a) the average number of taxis waiting?

(b) the proportion of customers finding a taxi when they arrive?

Exercise 5. *Customers arrive to a shop according to a Poisson process N with parameter λ. Customers are served one by one, in the order in which they arrived, and the service times Z_i for each customer i follow i.i.d. exponential random variables with parameter μ, which are independent from N. We denote X_t the number of customers in the line at time $t \geq 0$, $S_1, S_2, ...$ the jump times of X (i.e. $S_0 := 0$, $S_n := \min\{t \geq S_{n-1} : S_t \neq S_{n-1}\}$, $n \geq 1$) as well as $\tau_i := S_i - S_{i-1}, n \geq 1$ the interarrival times.*

1. *Get convinced that $S_1, S_2, ...$ are the jump times of N. What is the (joint) distribution of $\tau_1, \tau_2, ...$? What is the distribution of S_n, $n \geq 0$?*
2. *What is the distribution of τ_{n+1} given $X_{S_n} = 0$?*
3. *What is the distribution of τ_{n+1} given $X_{S_n} > 0$?*
4. *What is the distribution of $X_{S_{n+1}}$ given X_{S_n}?*
5. *What is the transition matrix $Q = (q_{ij})_{i,j}$ associated with X (as defined in Problem 1)? What are the transition rates $p_{i,j}$ of X, $\imath \neq j$? What is the generator R of the continuous-time Markov chain X?*
6. *Under which condition is the continuous-time Markov chain X transient? recurrent? positive recurrent?*
7. *Show that for every $C > 0$, $\pi_k := C(\frac{\lambda}{\mu})^n$ is a stationary measure, i.e. $\pi \geq 0$ and $\pi P = P$.*
8. *Does X have a stationary distribution (i.e. a stationary measure π with $\sum_{k \geq 1} \pi_k = 1$)?*
9. *Does X have a limiting distribution?*