Exercises # 8: Continuous-time Markov chains

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November 14, 2019

Due at 12:40pm on **Friday, November 15**. Python codes must be sent by email before due time. Each problem is worth 10 points, please do all problems.

Exercise 1. *Give an example of a continuous-time Markov chain X with more than one state, and explain why it is a continuous-time Markov chain. What is the expected time before the first transition of X occur? What is the transition matrix Q associated with X? (Recall that $q_{i,j}$ is the probability that X, starting in state i, will make its next transition to state j). What is the asymptotic behavior of the process (long-run proportion of time spent in each state, limiting distribution)?*

Bonus: write a Python code to simulate 10 paths of this continuous time Markov chain on a well-chosen time interval, and plot them on the same figure. Plot also as a function of time the proportions of time spent in each state.

Solution 1. *We have seen that a continuous time Markov chain can be defined as a process X such that, if it is at any time t in state i, it will remain in state i for a time $\tau_i \sim \exp(\lambda_i)$, and at time $t + \tau_i$ will make a transition to a state j according to Markov transition probabilities $Q = (q_{i,j})_{i,j}$. Thus a continuous time Markov chain is determined by a family of non-negative rates $(\lambda_i)_{i\geq 0}$ and a Markov transition matrix Q. The simplest continuous time Markov chain with more than one state is a two-state continuous time Markov chain, which we have studied in class: the only 2×2 matrix Q with $q_{i,i} = 0$ and $\sum_j q_{i,j} = 1$ is*

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus the distribution of a two-state continuous time Markov chain is entirely determined by the transition rates λ_0 and λ_1 respectively from states 0 and 1. We have also seen that the R matrix is

$$R = \begin{bmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{bmatrix}$$

Since the λs are bounded by their maximum, the process has no explosion; $R\pi = 0$ has a unique solution with $\pi_0 + \pi_1 = 1$, given by $\pi = \left(\frac{\lambda_1}{\lambda_0+\lambda_1}, \frac{\lambda_1}{\lambda_0+\lambda_1}\right)$. Thus this chain is ergodic, and converges to π in distribution by the ergodic theorem; the long-run proportion of time spent is each state is also given by π.

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Exercise 2. *Please do Problem 5 page 412 in the textbook.*

Solution 2.

(a) *Yes: the number of individuals infected between time s and t depends in the past only through the number of persons infected at time s.*

(b) *This is a pure birth process (transitions are only possible from states n to $n+1$, $n \geq 0$).*

(c) *Let S_i be the time of the i^{th} infection. $\mathbb{E}[S_i - S_{i-1}] = \frac{1}{\lambda_i}$, where λ_i is the rate of infection given that i individual are currently infected. Thus $\mathbb{E}[S_N] = \sum_{i=1}^{N-1} \frac{1}{\lambda_i}$.*

Given that a contact occur in a population with i infected individuals and $N-i$ non-infected individuals, it will lead to the infection of a new individual with probability $p\frac{i(N-i)}{\binom{2}{N}}$, yielding $\lambda_i := 2\lambda p \frac{i(N-i)}{N(N-1)}$ and

$$\mathbb{E}[S_N | X_0 = 1] = \frac{1}{2\lambda p} \sum_{i=1}^{N-1} \frac{N(N-1)}{i(N-i)}$$

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Exercise 3. *Please do Problem 9 page 413 in the textbook.*

Solution 3. *Since the death rate is constant, it follows that as long as the system is not empty (not is state 0) the number of deaths in any interval of length t will be a Poisson random variable with mean μt, from which*

$$p_{i,j}(t) := P(X_t = j | X_0 = i) = \begin{cases} 0 \quad \textit{if } i < j \\ e^{-\mu t} \frac{(\mu t)^{i-j}}{(i-j)!} \quad \textit{if } 0 < j \leq i \\ e^{-\mu t} \sum_{k \geq i} \frac{(\mu t)^k}{k!} \quad \textit{if } j = 0 \end{cases}$$

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Exercise 4. *Customers and taxis arrive to a taxi station according to independent Poisson processes with respective rates of two and three per minute. Taxis wait regardless of the number of other taxis present. However, a customer who does not find any taxi waiting when he arrives leaves immediately. What are:*

(a) *the average number of taxis waiting?*

(b) *the proportion of customers finding a taxi when they arrive?*

Solution 4. *See Exercise 24 of the textbook for the classical version of this classical problem.*

(a) *The number of taxis waiting (let us call it X) is a birth and death process with constant birth and death rates $\lambda_n = \lambda = 3$, $\mu_n = \mu = 2$; its mean satisfies*

$$M(t+h) - M(t) = (\lambda - \mu)h + o(h),$$

whence $M'(t) = (\lambda - \mu)$ and $M(t) = M(0) + (\lambda - \mu)t = t$: taxis will almost surely queue up towards an infinite line.

(b) *At any time t the proportion of customers finding a taxi waiting at their arrival is $(1 - p_0(t))$, where $p_0(t) = P(X_t = 0|X_0 = 0)$ is the probability that the birth and death process X is in state 0 at time t. Computing explicitly $p_0(t)$ is slightly delicate, however for $t \geq 1$ we have*

$$\frac{X_t}{t} = \frac{1}{t}\sum_{k=1}^{[t]-1}(X_{k+1} - X_k) + (X_t - X_{[t]}) \xrightarrow[t\to\infty]{a.s.} (\lambda - \mu)$$

which shows that $X_t \xrightarrow[t\to\infty]{a.s.} \infty$; in particular $p_0(t) = p(X_t = 0) \xrightarrow[t\to\infty]{} 0$ and the proportion of customers finding a taxi waiting at their arrival converges to 1.

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Exercise 5. *Customers arrive to a shop according to a Poisson process N with parameter λ, which jump times we denote $S_1, S_2, ...$ (i.e. $S_0 := 0$, $S_n := \min\{t \geq S_{n-1} : X_t \neq X_{S_{n-1}}\}$, $n \geq 1$) and interarrival times $\tau_i := S_i - S_{i-1}, n \geq 1$. Customers are served one by one, in the order in which they arrived, and the service times Z_i for each customer i follow i.i.d. exponential random variables with parameter μ, which are independent from N. We denote X_t the number of customers in the line at time $t \geq 0$.*

1. *What is the (joint) distribution of $\tau_1, \tau_2, ...$? What is the distribution of S_n, $n \geq 0$?*
2. *What is the distribution of τ_{n+1} given $X_{S_n} = 0$?*
3. *What is the distribution of τ_{n+1} given $X_{S_n} > 0$?*
4. *What is the distribution of $X_{S_{n+1}}$ given X_{S_n}?*
5. *What is the transition matrix $Q = (q_{ij})_{i,j}$ associated with X (as defined in Problem 1)? What are the transition rates $p_{i,j}$ of X, $\imath \neq j$? What is the generator R of the continuous-time Markov chain X?*

6. *Under which condition is the continuous-time Markov chain X transient? recurrent? positive recurrent?*

7. *Show that for every $C > 0$, $\pi_k := C(\frac{\lambda}{\mu})^n$ is a stationary measure, i.e. $\pi \geq 0$ and $\pi P = P$.*

8. *Does X have a stationary distribution (i.e. a stationary measure π with $\sum_{k\geq 1} \pi_k = 1$)?*

9. *Does X have a limiting distribution?*

Solution 5.

0. *The jump times of X are the jump times of N as well as the times at which services end! Let us denote $0 = \widetilde{S}_0$ as well as $\widetilde{S}_1, \widetilde{S}_2, \ldots$ the jump times of X, and $\tau_i := S_i - S_{i-1}, n \geq 1$ its interarrival times.*

1. *$\tau_1, \tau_2, \ldots$ are i.i.d. exponential random variables with parameter λ. As the sum of n i.i.d. exponential random variables with parameter λ, S_n has a gamma distribution with parameters n and λ.*

2-3. *τ_{n+1} is independent from X_{S_n}, thus from the events $X_{S_n} = 0$ and $X_{S_n} > 0$. More interesting is the distribution of $\widetilde{\tau}_{n+1}$ given $X_{\widetilde{S}_n}$.*

- *Given $X_{\widetilde{S}_n} = 0$, $\widetilde{\tau}_{n+1}$ is the time of the next arrival, namely an exponential random variable with parameter λ.*
- *Given $X_{\widetilde{S}_n} > 0$, $\widetilde{\tau}_{n+1}$ is the time of the next arrival or the time at which the service of the customer being currently served will be finished, whichever comes first. This is the minimum of two independent exponential random variables with respective parameters λ and μ, hence an exponential random variable with parameter $\lambda + \mu$.*

4. *Here again the quantities of interest are the $X_{\widetilde{S}_n}$s. It follows from the preceding questions that $X_{\widetilde{S}_{n+1}}$ is the one step transition from $X_{\widetilde{S}_n}$ with the matrix*

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & \ldots \\ q & 0 & p & \ddots & & \vdots & \\ 0 & q & \ddots & \ddots & \ddots & \vdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \\ \vdots & & \ddots & \ddots & \ddots & p & \ddots \\ 0 & \ldots & \ldots & 0 & q & 0 & \ddots \\ \vdots & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

where $p := \frac{\lambda}{\lambda+\mu} =: 1 - q$.

5. *The transition matrix associated with X is precisely the matrix Q in the preceding question. The transition rate from state i to state j are $v_i q_{i,j}$ where $v_0 = \lambda$ and $v_i = \lambda + \mu$ ($i \neq 0$). Thus the generator R of the continuous-time Markov chain X is*

$$R = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots & 0 & \dots \\ \mu & -(\lambda+\mu) & \lambda & \ddots & & \vdots & \\ 0 & \mu & \ddots & \ddots & \ddots & \vdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \\ \vdots & & \ddots & \ddots & \ddots & \lambda & \ddots \\ 0 & \dots & \dots & 0 & \mu & -(\lambda+\mu) & \ddots \\ \vdots & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

6. *This is an irreducible Markov chain, so that all states share the same nature: transient, null recurrent or positive recurrent. We have seen in Problem 4 that, when $\lambda > \mu$, then $X \to \infty$; in particular all states are transient is this case.*

   In fact, transience and recurrence of the states of the continuous time Markov chain are properties of the associated discrete time Markoc chain, i.e. of the matrix Q. Thus we know directly from our study of the discrete time random walks that the states of X are recurrent if and only if $p \leq q$. The next questions will show that if $\lambda < \mu$, the chain is positive recurrent (because X has a stationary distribution); and if $\lambda = \mu$ the chain is null recurrent (because X does not have a stationary distribution).

7. *Plugging in $\pi_k := C(\frac{\lambda}{\mu})^n$ shows that $\pi R = 0$ (i.e. $\pi V Q = \pi$) for any $\lambda, \mu > 0$. Conversely any family $(\pi_j)_{j \geq 0}$ satisfying $\pi R = 0$ satisfies $\lambda \pi_0 = \mu \pi_1$ and*

$$\lambda \pi_{j-1} + \mu \pi_{j+1} = (\lambda + \mu) \pi_j$$

   for $j \geq 1$, which non-negative solutions are of the form $C(\frac{\lambda}{\mu})^n, C \geq 0$ when $\lambda \neq \mu$, and $(C + C'n)$ when $\lambda = \mu$ (see the classical theory of second order linear sequences, e.g. MIT Course notes).

8. *The chain will have a stationary distribution if and only if we can choose C such that $\sum_j \pi_j = 1$. This is possible if and only if $\lambda < \mu$, with in that case*

$$\pi_j = \frac{1}{\mu - \lambda} (\frac{\lambda}{\mu})^j.$$

9. *As events occur at rate bounded by $\lambda + \mu < \infty$ the continuous time Markov chain X has no explosion; additionally we have just shown that when $\lambda < \mu$, there exist a unique stationary distribution. Thus when $\lambda < \mu$, X is ergodic and converges to its stationary distribution. When $\lambda \geq \mu$, X converges to ∞ a.s.*