STT886 : Midterm # 1: Markov Chains

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Each problem is worth 10 points, and 40 points gives a full grade – you can do as many problems as you wish, but points above 40 will count only towards personal satisfaction.

Exercise 1 (MCQ)**.** *Right answer = +1 point; wrong answer = -1 point; blank answer = 0 points; however, the total grade for the problem cannot be lower than 0.*

Let S be a one dimensional simple random walk on $\mathbb{Z}$, i.e. $S_n := \sum_{i=1}^n Z_i, n = 1, 2, ...$ where Z_i are i.i.d. random variables taking values -1 and $+1$ with probability 0.5. Are the following stochastic processes Markov chains? Please write Yes, No or leave blank.

(i) $(S_n)_{n\geq 0}$ ----------

(ii) $(S_n + n)_{n\geq 0}$ ----------

(iii) $(S_n + n^2)_{n\geq 0}$ ----------

(iv) $(S_n + 10^n)_{n\geq 0}$ ----------

(v) $(S_n + (-1)^n)_{n\geq 0}$ ----------

(vi) $(|S_n|)_{n\geq 0}$ ----------

(vii) $(S_n^2 - n)_{n\geq 0}$ ----------

(viii) $(S_{2n})_{n\geq 0}$ ----------

(ix) $(\sum_{k=0}^n S_k)_{n\geq 0}$ ----------

(x) $\big((-1)^n \cos(\frac{n\pi}{2019})\big)_{n\geq 0}$ ----------

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Exercise 2. *Give the transition matrix of a five-state Markov chain of your choice such that*

1. *There are exactly two communication classes*
2. *Exactly one of the two classes is recurrent*
3. *The recurrent class is aperiodic*
4. *There are no absorbing states*
5. *Exactly one state in the recurrent class is accessible from a state in the transient class.*

Does this Markov chain have a stationary distribution? If yes, is it unique? Does it have a limiting distribution?

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Exercise 3. *Consider a gambler starting with an initial wealth of* $\$x$, $x \in \mathbb{N}$, *and then placing successively independent bets, at each of which he wins or loses* $\$1$ *with probabilities* p *and* $q := 1 - p$, *respectively. The gambler stops playing when (if and only if) his wealth reaches* $\$0$ *or* $\$y$ *for a given* $y \in \mathbb{N}$, $y > x$. *Let* S_n *denote the total wealth after the* n^{th} *bet,* $T := T(x, y) := \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = y\}$ *the time when the game stops, and let* $\phi(x) := \phi(x, y) := P(S_T = y)$, *i.e. the probability that the gambler doesn't end the game ruined, starting with* $\$x$.

1. *Argue that* S *is a Markov chain and give its state space and transition probabilities. Describe the communication classes, their nature (recurrent or transient) and periodicity.*
2. *Find a recursive formula for* $\phi(x)$ *by conditioning with respect to the outcome of the first step of the Markov chain.*
3. *If* $p = q = \frac{1}{2}$, *show that* $\phi(x) = \frac{x}{y}$.
4. *If* $p = q = \frac{1}{2}$, *does the Markov chain* S *have any stationary distribution? any limiting distribution?*
5. *If* $p = q = \frac{1}{2}$ *and* $x = 3$ *and* $y = 4$; *what is the expected number of time periods the gambler's wealth be exactly* $\$1$*?*

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Exercise 4. *Let* X_n *be the discrete time stochastic process defined by* $X_0 = 1$, $X_{n+1} = \sum_{i=1}^{X_n} Z_i^{(n+1)}$, $n \geq 0$, *where* $Z_i^{(n)}$, $i \geq 1$, $n \geq 1$, *are i.i.d. non-negative integer random variables with expectation* μ *and variance* σ^2.

(i) Is X *a Markov chain? Justify your answer.*

(ii) What is $\mathbb{E}[X_{n+1}|X_n]$, $n \geq 1$ *?*

(iii) Deduce a general expression for $\mathbb{E}[X_n]$, $n \geq 1$.

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Exercise 5. *Let* S *be the "generalized" random walk on the signed integers* $\mathbb{Z}$ *defined by* $S_n := \sum_{i=1}^{n} Z_i, n \in \mathbb{N}$, *where* Z_i *are i.i.d. integer random variables with finite expectation. Argue that* X *is a Markov chain, and show that if* $\mathbb{E}[Z_1] \neq 0$ *then all states are transient.*