STT886 : Midterm # 1: Markov Chains

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Each problem is worth 10 points, and 40 points gives a full grade – you can do as many problems as you wish, but points above 40 will count only towards personal satisfaction.

Exercise 1 (MCQ)**.** *Right answer = +1 point; wrong answer = -1 point; blank answer = 0 points; however, the total grade for the problem cannot be lower than 0.*

Let S be a one dimensional simple random walk on $\mathbb{Z}$, i.e. $S_n := \sum_{i=1}^n Z_i, n = 1, 2, \ldots$ where Z_i are i.i.d. random variables taking values -1 and $+1$ with probability 0.5. Are the following stochastic processes Markov chains? Please write Yes, No or leave blank.

(i) $(S_n)_{n\geq 0}$ ----------

(ii) $(S_n + n)_{n\geq 0}$ ----------

(iii) $(S_n + n^2)_{n\geq 0}$ ----------

(iv) $(S_n + 10^n)_{n\geq 0}$ ----------

(v) $(S_n + (-1)^n)_{n\geq 0}$ ----------

(vi) $(|S_n|)_{n\geq 0}$ ----------

(vii) $(S_n^2 - n)_{n\geq 0}$ ----------

(viii) $(S_{2n})_{n\geq 0}$ ----------

(ix) $(\sum_{k=0}^n S_k)_{n\geq 0}$ ----------

(x) $\big((-1)^n \cos(\frac{n\pi}{2019})\big)_{n\geq 0}$ ----------

Solution 1 (MCQ)**.** *In each case, when the given process X is a Markov chain we give its transition probabilities $p_{i,j}$ which are non-zero, and when it is not a Markov chain we give a counter example where the Markov property is broken; when writing conditional probabilities we need to check that the conditioning event has positive probability).*

(i) $(S_n)_{n\geq 0}$ *Yes:* $p_{i,i+1} = p_{i,i-1} = \frac{1}{2}, i \in \mathbb{Z}$

(ii) $(S_n + n)_{n\geq 0}$ *Yes:* $p_{i,i} = p_{i,i+2} = \frac{1}{2}, i \in \mathbb{Z}$

(iii) $(S_n + n^2)_{n\geq 0}$ *No:*

$$\frac{1}{2} = p(X_1 = 0|X_0 = 0) \neq p(X_2 = 0|X_1 = 0, X_0 = 0) = 0,$$

with

$$P(X_1 = 0, X_0 = 0) \geq P(S_1 = -1, S_0 = 0) > 0.$$

(iv) $(S_n + 10^n)_{n\geq 0}$ *Yes:* $p_{10^n+k,10^{n+1}+k+1} = p_{10^n+k,10^{n+1}+k-1} = \frac{1}{2}, |k| \leq n$

(v) $(S_n + (-1)^n)_{n\geq 0}$ *Yes:*

$$p_{2i,2i-1} = p_{2i,2i-3} = \frac{1}{2}, p_{2i+1,2i+2} = p_{2i+1,2i+4} = \frac{1}{2}, i \in \mathbb{Z}.$$

(vi) $(|S_n|)_{n\geq 0}$ *Yes:* $p_{0,1} = 1, p_{i,i-1} = p_{i,i-1} = \frac{1}{2}, i \geq 0.$

(vii) $(S_n^2 - n)_{n\geq 0}$ *No:*

$$\begin{gathered} 1 = P(X_1 = 0|X_0 = 0) \\ \neq P(X_5 = 0|X_0 = 0, X_1 = 0, X_2 = 2, X_3 = 6, X_4 = 0) \\ = P(X_5 = 0) = 0, \end{gathered}$$

with

$$P(X_0 = 0, X_1 = 0, X_2 = 2, X_3 = 6, X_4 = 0) \geq P(S_1 = 1, S_2 = 2, S_3 = 3, S_4 = 2) > 0.$$

(viii) $(S_{2n})_{n\geq 0}$ *Yes:* $p_{i,i} = 2p_{i,i+2} = 2p_{i,i-2} = \frac{1}{2}$

(ix) $(\sum_{k=0}^{n} S_k)_{n\geq 0}$ *No:*

$$\frac{1}{2} = \mathbb{P}(X_3 = 0|X_2 = 1, X_1 = 1, X_0 = 0) \neq \mathbb{P}(X_2 = 0|X_1 = 1, X_0 = 0) = 0,$$

with

$$P(X_2 = 1, X_1 = 1, X_0 = 0) \geq P(S_2 = 0, S_1 = 1, S_0 = 0) > 0.$$

(x) $\left((-1)^n\cos(\frac{n\pi}{2019})\right)_{n\geq 0}$ *Yes, of course:* $p_{(-1)^i\cos(\frac{i\pi}{2019}),(-1)^{i+1}\cos(\frac{(i+1)\pi}{2019})} = 1, i = 0, 2, ..., 2018$

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Exercise 2. *Give the transition matrix of a five-state Markov chain of your choice such that*

1. *There are exactly two communication classes*
2. *Exactly one of the two classes is recurrent*
3. *The recurrent class is aperiodic*
4. *There are no absorbing states*
5. *Exactly one state in the recurrent class is accessible from a state in the transient class.*

Does this Markov chain have a stationary distribution? If yes, is it unique? Does it have a limiting distribution?

Solution 2. *We can consider the following transition matrix:*

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

There are two communication classes, $T = \{1, 2, 3\}$ and $R = \{4, 5\}$. T is transient and has period 2, R is recurrent and aperiodic; exactly one state in R (state 4) is accessible from a state in T (state 3). since there is a finite number of states R is positive recurrent, and thus the Markov chain restricted to the recurrent class R has a unique stationary distribution, $\pi_4 = \pi_5 = \frac{1}{2}$. The stationary distribution of the whole chain can only attribute weights to recurrent states, so it is given by $(0, 0, 0, \frac{1}{2}, \frac{1}{2})$. After spending a finite time in the transient states the Markov chain converges to its stationary distribution (i.e. the stationary distribution is also a limiting distribution).

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Exercise 3. *Consider a gambler starting with an initial wealth of* $\$x$, $x \in \mathbb{N}$, *and then placing successively independent bets, at each of which he wins or loses* $\$1$ *with probabilities* p *and* $q := 1 - p$, *respectively. The gambler stops playing when (if and only if) his wealth reaches* $\$0$ *or* $\$y$ *for a given* $y \in \mathbb{N}$, $y > x$. *Let* S_n *denote the total wealth after the* n^{th} *bet,* $T := T(x, y) := \inf\{n \geq 0 : S_n = 0$ *or* $S_n = y\}$ *the time when the game stops, and let* $\phi(x) := \phi(x, y) := P(S_T = y)$, *i.e. the probability that the gambler doesn't end the game ruined, starting with* $\$x$.

1. *Argue that* S *is a Markov chain and give its state space and transition probabilities. Describe the communication classes, their nature (recurrent or transient) and periodicity.*

2. *Find a recursive formula for* $\phi(x)$ *by conditioning with respect to the outcome of the first step of the Markov chain.*

3. *If* $p = q = \frac{1}{2}$, *show that* $\phi(x) = \frac{x}{y}$.

4. *If* $p = q = \frac{1}{2}$, *does the Markov chain* S *have any stationary distribution? any limiting distribution?*

5. *If* $p = q = \frac{1}{2}$ *and* $x = 3$ *and* $y = 4$; *what is the expected number of time periods the gambler's wealth be exactly* $\$1$?

Solution 3.

1-4. *See the textbook and HW4 for this classical problem.*

5. *The question is about the expected time spent in a transient state. With* $y = 4$ *we have 4 states* $\{0, 1, 2, 3\}$ *where the states* $0, 4$ *are recurrent and* $1, 2$ *transient. The matrix giving the transitions between transient states is*

$$P_T = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

so that

$$S_T := (I_T - P_T)^{-1} = \frac{4}{3} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}.$$

With $x = 2$, *the gambler will spend in average* $s_{2,1} = \frac{2}{3}$ *time periods in the transient state 1 corresponding to a current wealth of* $\$1$.

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Exercise 4. *Let X_n be the discrete time stochastic process defined by $X_0 = 1$, $X_{n+1} = \sum_{i=1}^{X_n} Z_i^{(n+1)}$, $n \geq 0$, where $Z_i^{(n)}$, $i \geq 1$, $n \geq 1$, are i.i.d. non-negative integer random variables with expectation μ and variance σ^2.*

(i) Is X a Markov chain? Justify your answer.

(ii) What is $\mathbb{E}[X_{n+1}|X_n]$, $n \geq 1$?

(iii) Deduce a general expression for $\mathbb{E}[X_n]$, $n \geq 1$.

Solution 4. *See the textbook and lecture notes for this classical problem.*

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Exercise 5. *Let S be the "generalized" random walk on the signed integers $\mathbb{Z}$ defined by $S_n := \sum_{i=1}^{n} Z_i, n \in \mathbb{N}$, where Z_i are i.i.d. integer random variables with finite expectation. Argue that X is a Markov chain, and show that if $\mathbb{E}[Z_1] \neq 0$ then all states are transient.*

Solution 5. *By the strong law of large numbers $\frac{S_n}{n} \xrightarrow[a.s.]{n\to\infty} \mathbb{E}[Z_1]$. Thus if $\mathbb{E}[Z_1] \neq 0$, e.g. $\mathbb{E}[Z_1] > 0$, we have $S_n \xrightarrow[a.s.]{n\to\infty} +\infty$ a.s. This means that S will spend a finite number of steps in any finite subset of states: since every state is contained in a finite subset of states, all states are transient.*