STT886 : Midterm # 2: Point processes

Léo Neufcourt

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Please do any four problems. If your choice is ambiguous the first four problems found will be graded.

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Exercise 1 (MCQ)**.** *Please select the best answer. Correct = +1 point; incorrect = -0.5 point; blank = 0 points; however, the total grade for the problem cannot be lower than 0.*

1. *A Poisson process is*

   (a) *a renewal process* (b) *a continuous-time Markov chain*

   (c) *both* (d) *none*

2. *An irreducible discrete-time Markov chain on a finite state space*

   (a) *always has a stationary distribution* (b) *always has a limiting distribution*

   (c) *both* (d) *none*

3. *The times between successive jumps of a continuous-time Markov chain are*

   (a) *exponential random variables* (b) *i.i.d. random variables*

   (c) *both* (d) *none*

4. *The times between successive jumps of a regenerative process are*

   (a) *exponential random variables* (b) *i.i.d. random variables*

   (c) *both* (d) *none*

5. *Every irreducible continuous-time Markov chain on a finite state space has a limiting distribution*

   (a) .
   (b) *if it is positive recurrent*
   (c) *if it has no explosion*
   (d) *none of the above*

6. *Every birth and death process is a*

   (a) *continuous time Markov chain*
   (b) *counting process*
   (c) *both*
   (d) *none*

7. *A pure birth process with birth rates* $\lambda_n = 2n$ *starting with 1 individuals at time 0 has its expectation at time* 3 *given by*

   (a) 1
   (b) 2^3
   (c) e^6
   (d) ∞

8. *Considering a sequence of events occurring at a Poisson rate of 1 per hour, it is*

   (a) *more*
   (b) *less*
   (c) *equally*
   (d) *uncomparably*

   likely that the next event of the process will occur within the next hour than it will occur after one hour.

9. *Considering a renewal process* X *starting at 0 with integer interarrival times, if at any time* n *there is a probability* p^n *that a renewal occurs, then* X *has a mean function given by*

   (a) $m(n) = \frac{n}{1-p}$
   (b) $m(n) = \frac{p}{1-p}$
   (c) $m(n) = n\frac{p}{1-p}$
   (d) $m(n) = p^n$

10. *The class of*

    (a) *discrete-time Markov chains*
    (b) *continuous-time Markov chains*
    (c) *Markov processes*
    (d) *semi-Markov processes*

    is the smallest class of stochastic processes containing both continuous-time Markov chains and discrete-time Markov chain

Solution 1. *Note that two sets with different ordering were used.*
1 : c,
2 : a (aperiodicity is required for ergodicity and convergence),
3 : a ($\tau_i \sim \mathcal{E}(v_i)$)*,*
4 : d (the times between successive renewals *are i.i.d.),*
5 : b (a continuous-time Markov chain on a finite state space has no explosion and when it is irreducible has a (unique) stationary distribution distribution $(p_j)_j$*, obtained from the unique stationary distribution* π *of the embedded irreducible finite-state discrete-time Markov chain as*

$$p_j = \frac{1}{\sum_i \frac{\pi_i}{v_i}} \frac{\pi_j}{v_j}$$

where the sum is well-defined as all $v_j > 0$ *and finite),*
6 : a (a birth-and-death process can decrease),
7 : c,
8 : a ($P(X > 1) = e^{-1} < \frac{1}{2}$)*,*
9 : b ($m(n) = p\frac{1-p^n}{1-p} \xrightarrow[n\to\infty]{} \frac{p}{1-p}$)*,*
10 : d.

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Exercise 2. *Consider a gambler starting with an initial wealth of* $\$x$, $x \in \mathbb{N}$, *and then placing successively independent bets, at each of which he wins or loses* $\$1$ *with probabilities* p *and* $q := 1-p$*, respectively. The gambler stops playing when (if and only if) his wealth reaches* $\$0$ *or* $\$y$ *for a given* $y \in \mathbb{N}$, $y > x$*. Let* S_n *denote the total wealth after the* n^{th} *bet,* $T := T(x, y) := \inf\{n \geq 0 : S_n = 0 \text{ or } S_n = y\}$ *the time when the game stops, and let* $\phi(x) := \phi(x, y) := P(S_T = y)$*, i.e. the probability that the gambler doesn't end the game ruined, starting with* $\$x$*.*

1. *Argue that* S *is a Markov chain and give its state space and transition probabilities. Describe the communication classes, their nature (recurrent or transient) and periodicity.*

2. *Show that* $\phi(x)$ *satisfies*

$$q\phi(x-1) + p\phi(x+1) = \phi(x), x \geq 1$$

   and deduce the general expression of $\phi(x)$ *when* $p = q$ *and* $p \neq q$*.*

3. *What is* $\mathbb{E}[T]$*, when* $p \neq q$*?*

4. *Consider a population described by a birth and death process, with constant birth and death rates respectively* $\lambda = 10$ *and* $\mu = 1$*, starting with one individual. What is the probability that the population will reach 10 (individuals) before reaching 0?*

Solution 2. *1-2. See the textbook, Homework 4 and Midterm 1. Denoting* $\rho := \frac{q}{p}$*, we have we have*

$$\phi(x,y) = \frac{1-(\frac{q}{p})^x}{1-(\frac{q}{p})^y} = \frac{1-\rho^x}{1-\rho^y}$$

when $q \neq p$*, and* $\phi(x,y) = \frac{x}{y}$ *when* $q = p$*.*

3. $T(x,y)$ *is the first time at which* $\sum_{i=1}^n X_i = (y-x)$ *or* $-x$*, where* X_i *are i.i.d. random variables taking values* ± 1 *with respective probabilities* p *and* q*.* $T(x,y)$ *is a stopping time, so Wald's equation holds:*

$$\mathbb{E}[\sum_{i=1}^T X_i] = \mathbb{E}[X_1]\mathbb{E}[T].$$

On the one hand,

$$\mathbb{E}[\sum_{i=1}^T X_i] = (y-x)\phi(x,y) - x(1-\phi(x,y)) = y\phi(x,y) - x;$$

on the other hand

$$\mathbb{E}[X_1] = (p-q) = \frac{1-\rho}{1+\rho}$$

after using $q = p\rho$ *and* $p = \frac{1}{1+\rho}$*. Thus we can conclude*

$$\mathbb{E}[T(x,y)] = \frac{1+\rho}{1-\rho}(y\frac{1-\rho^x}{1-\rho^y} - x).$$

4. This probability is the same as the probability that a gambler starting at $x = 1$ *will reach* $y = 10$ *before* 0*, when placing independent successive bets, each of which with success probability* $p = \frac{\lambda}{\lambda+\mu} = \frac{10}{11}$*. Thus it is given by.*

$$\phi(1,10) = \frac{1-(\frac{1}{10})}{1-(\frac{1}{10})^{10}} = \frac{9}{10}\frac{1}{1-(\frac{1}{10})^{10}} \approx \frac{9}{10}$$

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Exercise 3. *In each of the following cases, does the* $\lim_{t\to\infty} X_t$ *exist, almost surely or in distribution? Please justify your answer and provide the limit, when it exists.*

(i) X *is a Poisson process with rate parameter* $\lambda > 0$

(ii) X *is a non-homogeneous Poisson process with rate function* $\lambda(t) = e^{-t}$

(iii) X *is an ergodic continuous-time Markov chain with stationary distribution* π

(iv) *X is a renewal process with expected interarrival time* $\mu > 0$

(v) *X is a reward renewal process with expected interarrival time* $\mu > 0$ *and expected rewards* $r < 0$

Solution 3.

(i) *The strong law of large number for the Poisson process (or renewal processes) states* $\frac{X_t}{t} \xrightarrow[t\to\infty]{a.s.} \lambda$, *thus* $X_t \xrightarrow[t\to\infty]{a.s.} \infty$ *(assuming* $\lambda > 0$*).*

(ii) *As we have seen in the homework* $P(X_t = n) \xrightarrow[t\to\infty]{a.s.} \frac{e^{-1}}{n!}$ *which means that*

$$X_t \xrightarrow[n\to\infty]{d} \mathcal{P}oisson(1).$$

(iii) *The ergodic theorem for continuous-time Markov chain establishes that*

$$X_t \xrightarrow[n\to\infty]{d} \pi.$$

(iv) *It follows from the strong law of large numbers for renewal processes that* $\frac{X_t}{t} \xrightarrow[n\to\infty]{a.s.} \frac{1}{\mu}$, *whence*

$$X_t \xrightarrow[t\to\infty]{a.s.} \infty.$$

(v) *It follows from the strong law of large numbers for reward renewal processes that* $\frac{X_t}{t} \xrightarrow[n\to\infty]{a.s.} \frac{r}{\mu}$, *whence*

$$X_t \xrightarrow[t\to\infty]{a.s.} -\infty.$$

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Exercise 4. *Customers and taxis arrive to a taxi station according to independent Poisson processes with respective rates of* $\mu = 3$ *and* $\lambda = 2$ *per minute. In each of the following scenarios, what are in the long run,*

(a) *the average number of taxis waiting?*

(b) *the proportion of customers finding a taxi when they arrive?*

1. *There are only two parking spaces, so taxis arriving to the station wait if there is no more than one taxi already waiting, and leave otherwise.*

2. *There are n parking spaces, so that taxis arriving to the station wait if there is no more than* $n-1$ *taxis already waiting, and leave otherwise.*

3. *There are infinitely many parking spaces, so that taxis arriving to the station wait regardless of the number of other taxis present.*

In all cases, a customer who does not find any taxi waiting when he arrives leaves immediately.

Solution 4. *Let us denote X the number of taxis waiting. In all three cases, X is clearly a continuous-time Markov chain. If it has stationary distribution π, then the expected number of taxis waiting in $x := \sum_{j=1}^{\infty} j\pi_j$ and the proportion of customers finding a taxi when they arrived is the proportion of time X is not is state 0, i.e. $\rho := 1 - \pi_0$.*

1. *There are here three states: 0, 1 and 2. The stationary distribution must satisfy $\pi_j = \sum_i \pi_i v_i q_{ij} = 0$. which writes*

$$\begin{cases} \lambda\pi_0 = \mu\pi_1 \\ (\lambda+\mu)\pi_1 = \lambda\pi_0 + \mu\pi_2 \\ \mu\pi_2 = \lambda\pi_1 \end{cases}$$

The unique normalized solution is $\pi_1 = \frac{\lambda}{\mu}\pi_0$, $\pi_2 = (\frac{\lambda}{\mu})^2\pi_0$ and

$$\pi_0 = \frac{1}{1 + (\frac{\lambda}{\mu}) + (\frac{\lambda}{\mu})^2} = \frac{9}{19}.$$

Hence $x = \pi_1 + 2\pi_2 = \frac{14}{19}$ and $\rho = 1 - \pi_0 = \frac{10}{19}$.

2. *The same argument (see also the textbook) leads to $\pi_k = (\frac{\lambda}{\mu})^k\pi_0, 0 \le k \le n$ and*

$$\pi_0 = \frac{1}{1 + (\frac{\lambda}{\mu}) + (\frac{\lambda}{\mu})^2 + \ldots + (\frac{\lambda}{\mu})^n} = \frac{1 - \frac{\lambda}{\mu}}{1 - (\frac{\lambda}{\mu})^{n+1}}.$$

3. *With $\lambda < \mu$, the same argument (see also the textbook, homework) leads to*

$$\pi_n = (\frac{\lambda}{\mu})^k\pi_0, n \ge 0$$

and

$$\pi_0 = \frac{1}{1 + (\frac{\lambda}{\mu}) + (\frac{\lambda}{\mu})^2 + \ldots + (\frac{\lambda}{\mu})^n + \ldots} = 1 - \frac{\lambda}{\mu}.$$

This yields $x = 2$ and $\rho = \frac{2}{3}$.

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Exercise 5. *An (innocent) prisoner is trying to escape his prison cell. This is a large prison (we suppose that it has an infinite number of cells) and many tunnels have been dug over the years by prisoners: each cell has access to two tunnels, one leading to the guards' room, and one leading to the prison yard. Both tunnels take an average time of one hour to be crossed. If the prisoner arrives in the guards' room, (s)he is taken to a new cell after being watched carefully for one day. If (s)he reaches the yard, the prisoner tries to jump over the fence; this attempt takes an average 30 minutes, and (s)he succeeds with probability $\frac{1}{2}$; else (i.e. with probability $\frac{1}{2}$) (s)he is caught and taken to a new cell after being watched carefully for one day. What is the probability that the prisoner will escape? How much time will that take in average?*

Solution 5. *Denote 0, 1 and 2 the states respectively corresponding to the prisoner being in the cell, the yard and free. The location of the prisoner over time forms a semi-Markov process with non-zero transition probabilities $p_{0,1} = 1$, $p_{1,0} = p_{1,2} = \frac{1}{2}$, $p_{2,2} = 1$ and average waiting times μ_0, μ_1 and μ_2, all positive. Recurrence is a property shared by all semi-Markov processes based on the same transition matrix, so 2 is the unique recurrent state: this implies that prisoner will escape with probability 1, and that the time T at which the prisoner will escape is finite almost surely. Now, by conditioning on the outcome of the first escape attempt we obtain*

$$\mathbb{E}[T] = 1 + \frac{1}{2}(24 + \mathbb{E}[T]) + \frac{1}{2}[\frac{1}{2} + \frac{1}{2}(24 + \mathbb{E}[T])],$$

yielding $\mathbb{E}[T] = 77$ hours. Note that the waiting times can be expressed explicitly: indeed, $\mu_0 = 1 + \frac{1}{2}(24 + \mu_0)$, yielding $\mu_0 = 26$; $\mu_1 = \frac{1}{2} + \frac{1}{2}24 = 12.5$; and $\mu_2 = \infty$.